Critical lattice size limit for synchronized chaotic state in one- and two-dimensional diffusively coupled map lattices

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We consider diffusively coupled map lattices with P neighbors (where P is arbitrary) and study the stability of the synchronized state. We show that there exists a critical lattice size beyond which the synchronized state is unstable. This generalizes earlier results for nearest neighbor coupling. We confirm the analytical results by performing numerical simulations on coupled map lattices with logistic map at each node. The above analysis is also extended to two-dimensional P-neighbor diffusively coupled map lattices.

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I. INTRODUCTION

In recent years, synchronization of coupled dynamical systems [1–3] has become an important area of research for their applications in a variety of fields including secure communications, cryptography, optics, neural networks, pattern formation, geophysics, and population dynamics [4–7]. In particular, the stability of synchronized state in coupled map lattices (CML) with various coupling schemes have been studied extensively [8–23]. To be specific the CML with diffusive coupling has attracted considerable attention in recent studies. In such systems, the synchronized state is not stable when the number of nodes exceeds a certain critical limit and each node is coupled only with its nearest neighbor [11,13]. In this paper, using the formalism put forth in Refs. [11,13] we derive an exact analytic expression for this limit for a more general case of P-neighbor coupling. Further, the results are verified through numerical simulations in coupled logistic map lattices. All the analyses are carried out in both one-dimensional (1D) and two-dimensional (2D) CMLs. Studies similar to our present work, but for coupled oscillators, are reported in Refs. [24,25].

II. CRITICAL SIZE LIMIT IN 1D CASE

Consider one-dimensional coupled map lattices with P-neighbor diffusive coupling represented by

$$\mathbf{x}_j(n+1) = f(\mathbf{x}_j(n)) + \frac{1}{2P}\sum_{p=1}^{P} a_p[f(\mathbf{x}_{j-p}(n)) + f(\mathbf{x}_{j+p}(n)) - 2f(\mathbf{x}_j(n))], \quad (1)$$

where $\mathbf{x}_j$ is a M-dimensional state vector, j represents the lattice site, L is the lattice size, a_p is the coupling strength between the jth map and its pth neighbor, and the evolution of the map at the jth site is described by $f(\mathbf{x}_j(n))$. Also the periodic boundary condition is imposed and the synchronized state (synchronization manifold) is defined by $\mathbf{x}_1(n)=\mathbf{x}_2(n)=\cdots=\mathbf{x}_L(n)=\mathbf{x}(n)$. Since a_p is a very general coupling coefficient, the long range model proposed by Antenedo [8] can be incorporated into the above equation.

Linearizing Eq. (1) around $\mathbf{x}$ and performing the discrete spatial Fourier transform $\eta_l(n)=\frac{1}{L}\sum_{j=1}^{L}\exp(-i2\pi jl/L)\mathbf{z}_j(n)$, the resulting form after simplification (see Refs. [6,12] for details) is

$$\mu_i(l) = h_i + \ln\left|1 - \frac{2}{P}\sum_{p=1}^{P} a_p \sin^2(\pi pl/L)\right|, \quad i=1,2,\ldots,M, \quad l=0,1,\ldots,L-1. \quad (2)$$

Here $\mu_i(l)$'s are the Lyapunov exponents corresponding to the lth mode and h_i's are the Lyapunov exponents of the isolated map ordered as $h_1 \geq h_2 \geq \cdots \geq h_M$. The mode $l=0$ corresponds to the synchronized state and the other modes represent its transverse variations. Hence $\mu_1(l)$ gives the largest transverse Lyapunov exponent for the mode $l\neq 0$. Therefore the stability of the synchronized state is ensured if $\mu_1(l)<0$ for all $l\neq 0$. However, the symmetry in Fourier modes reduces this condition as $\mu_1(l)<0$ for $l=1,2,\ldots,L/2$ [$(L-1)/2$ if L is odd]. Thus the stability condition reduces to

$$\left|1 - \frac{2}{P}\sum_{p=1}^{P} a_p \sin^2(\pi pl/L)\right| < \exp(-h_1), \quad l=1,2,\ldots,L/2 \text{ or } (L-1)/2, \quad (3)$$

and this expression is also obtained in Ref. [6]. We use this condition to derive the expression for the critical lattice size limit in the rest of this paper.

Let $\lambda_l = 1-\frac{2}{P}\sum_{p=1}^{P}a_p\sin^2(\pi pl/L)$ and define $\lambda_{\max}=\max\{\lambda_l\}$, $\lambda_{\min}=\min\{\lambda_l\}$. Then the above stability condition can be rewritten as

$$\lambda_{\max} < \exp(-h_1), \quad \lambda_{\min} > -\exp(-h_1). \quad (4)$$

Let $\lambda_l^{\epsilon}=1-\frac{2\epsilon}{P}\sum_{p=1}^{P}\sin^2(\pi pl/L)$, $\epsilon=\min\{a_p\}$, and define an upper bound on $\lambda_{\max}$ as $\lambda_{\max}^{*}=\max\{\lambda_l^{\epsilon}\}$. Therefore the first stability condition is ensured if $\lambda_{\max}^{*}<\exp(-h_1)$. Similarly, let $\lambda_l^{\epsilon'}=1-\frac{2\epsilon'}{P}\sum_{p=1}^{P}\sin^2(\pi pl/L)$, $\epsilon'=\max\{a_p\}$, and define a lower bound on $\lambda_{\min}$ as $\lambda_{\min}^{*}=\min\{\lambda_l^{\epsilon'}\}$. Hence the second stability condition in Eq. (4) is ensured if $\lambda_{\min}^{*}>-\exp(-h_1)$.

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We start with the simplest case of nearest neighbor coupling for which results already exist [11,13] and then we extend the idea to a more general P-neighbor coupling. In the nearest neighbor coupling case, $\lambda^*_{max}=1-2\epsilon\sin^2(\pi/L)$. Therefore the first stability condition is satisfied if $\epsilon > \frac{1-\exp(-h_1)}{2\sin^2(\pi/L)}$, where $\epsilon \le a_p\ \forall\ p$. Consequently, in terms of the coupling coefficients, the above condition becomes

$$a_p > \frac{1-\exp(-h_1)}{2\sin^2(\pi/L)} \quad \forall\ p. \tag{5}$$

Similarly, $\lambda^*_{min}=1-2\epsilon$. Hence the second stability condition is satisfied if $\epsilon' < \frac{1+\exp(-h_1)}{2}$, where $\epsilon' \ge a_p\ \forall\ p$. In terms of coupling coefficients, we get

$$a_p < \frac{1+\exp(-h_1)}{2} \quad \forall\ p. \tag{6}$$

Combining the two inequalities (5) and (6), we get the final stability condition [11,13] as

$$\frac{1-\exp(-h_1)}{2\sin^2(\pi/L)} < a_p < \frac{1+\exp(-h_1)}{2} \quad \forall\ p. \tag{7}$$

As L becomes larger, the above stability range becomes smaller and at a particular critical value of L, the range shrinks to zero. Beyond this critical value of L, the stability condition (7) is violated and hence the synchronized state can never be stable. At this critical value of L, one can replace the inequalities by equality signs in Eq. (7) and get

$$L_{1,1} = \mathrm{Int}\left[\frac{\pi}{\sin^{-1}(\sqrt{\tanh(h_1/2)})}\right], \tag{8}$$

where $L_{1,1}$ is the maximum lattice size that can support synchronized chaos in a one-dimensional nearest neighbor diffusively coupled map lattice.

Let us now turn to the more general case of P-neighbors diffusively coupled map lattices for which no previous analytical results exist. However, similar results do exist for coupled oscillators [24,25]. After making use of some simple trigonometric relations, the expressions for λ_l^{ϵ} and $\lambda_l^{\epsilon'}$ take the forms

$$\lambda_l^{\epsilon} = 1-\epsilon\left[1-\frac{\sin(P\pi l/L)\cos[(P+1)\pi l/L]}{P\sin(\pi l/L)}\right], \tag{9}$$

and

$$\lambda_l^{\epsilon'} = 1-\epsilon'\left[1-\frac{\sin(P\pi l/L)\cos[(P+1)\pi l/L]}{P\sin(\pi l/L)}\right], \tag{10}$$

respectively, where $\epsilon \le a_p \le \epsilon'\ \forall\ p$. The expression inside the square brackets takes its lowest value when $l=1$ and it takes its highest value for the mode $l=l_h=\mathrm{Int}[L_{1,1}/2]$, for all values of P. Following the same procedure as in the nearest neighbor case, we finally get

$$\frac{1-\exp(-h_1)}{\left[1-\frac{\sin(P\pi/L)\cos[(P+1)\pi/L]}{P\sin(\pi/L)}\right]} < a_p < \frac{1+\exp(-h_1)}{\left[1-\frac{\sin(P\pi l_h/L)\cos[(P+1)\pi l_h/L]}{P\sin(\pi l_h/L)}\right]} \quad \forall\ p. \tag{11}$$

At the critical coupling strength ϵ_c ($a_p=\epsilon_c\ \forall\ p$) the extreme values of a_p coincide and the above expression becomes

$$\frac{\sin(P\pi/L_{1,P})\cos[(P+1)\pi/L_{1,P}]}{\sin(\pi/L_{1,P})} = P\left[1-\tanh(h_1/2)\times\left(1-\frac{\sin(P\pi l_h/L_{1,P})\cos[(P+1)\pi l_h/L_{1,P}]}{P\sin(\pi l_h/L_{1,P})}\right)\right], \tag{12}$$

where $l_h=\mathrm{Int}[L_{1,1}/2]$ and $L_{1,1}$ is given in Eq. (8). The critical lattice size limit $L_{1,P}$ is obtained by solving the above transcendental equation numerically. In a special case of $P=L/2$ [or $(L-1)/2$ for odd L] we get $h_1=2\tanh^{-1}(1)$. This result indicates that the synchronized state is always possible for globally coupled map lattices as long as $h_1<\infty$. The dependence of $L_{1,P}$ on the maximum Lyapunov exponent of the isolated map (h_1) and the number of neighbors coupled (P) are shown in Figs. 1(a) and 1(b). It is observed that $L_{1,P}$ increases almost linearly with P for a particular value of h_1, and decays with h_1 for a particular value of P. Also, all the results are confirmed numerically by considering the logistic map (defined by $x(n+1)=1-r[x(n)]^2$) at each node. The variations of the critical coupling strength ϵ_c and the critical size limit $L_{1,P}$ with P (for $r=1.9$) are shown in Figs. 1(c) and 1(d).

III. CRITICAL SIZE LIMIT IN 2D CASE

Now we consider two-dimensional coupled map lattices with P-neighbor diffusive coupling of the form

$$\begin{aligned}\mathbf{x}_{j,k}(n+1) = f(\mathbf{x}_{j,k}(n)) + \frac{1}{4P}\sum_{p=1}^{P}\{a_p[f(\mathbf{x}_{j-p,k}(n)) \\ + f(\mathbf{x}_{j+p,k}(n)) - 2f(\mathbf{x}_{j,k}(n))] \\ + b_p[f(\mathbf{x}_{j,k-p}(n)) + f(\mathbf{x}_{j,k+p}(n)) - 2f(\mathbf{x}_{j,k}(n))]\},\end{aligned} \tag{13}$$

where $\mathbf{x}_{j,k}$ is a M-dimensional state vector, (j,k) represents the lattice site, L is the lattice size, and a_p and b_p are the coupling strengths between the (j,k)th map and its pth neighbor along j and k directions, respectively.

In this case, the stability condition for the synchronized state is

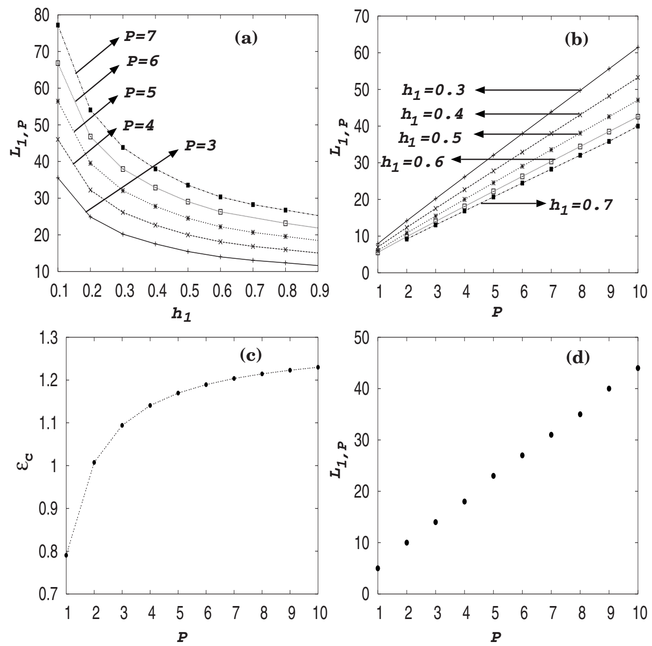

FIG. 1. (a) The variation of $L_{1,P}$ with h_1, (b) the variation of $L_{1,P}$ with P, (c) the variation of ϵ_c with P in one-dimensional coupled logistic map lattices, obtained for $r=1.9$, and (d) the variation of $L_{1,P}$ with P in one-dimensional coupled logistic map lattices, obtained for $r=1.9$ ($h_1=0.5554$).

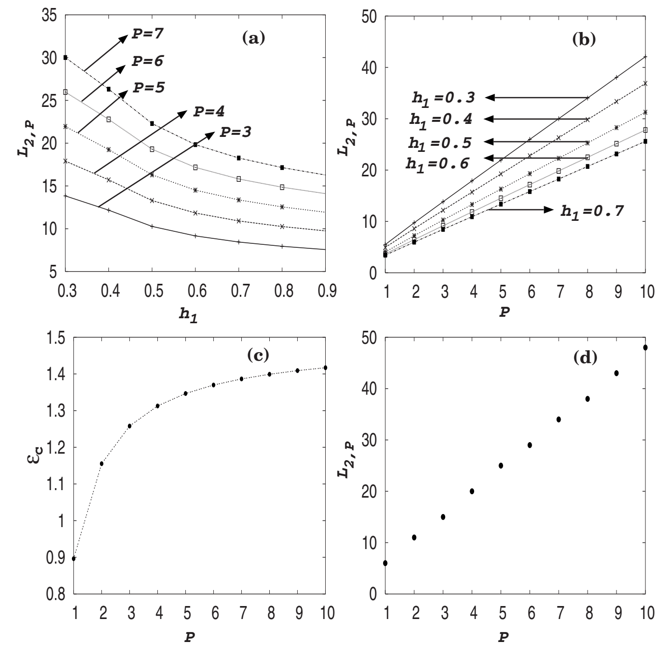

FIG. 2. (a) The variation of $L_{2,P}$ with h_1, (b) the variation of $L_{2,P}$ with P, (c) the variation of ϵ_c with P in two-dimensional coupled logistic map lattices, obtained for $r=1.5$, and (d) the variation of $L_{2,P}$ with P in two-dimensional coupled logistic map lattices, obtained for $r=1.5$ ($h_1=0.2378$).

$$\left| 1 - \frac{1}{P}\sum_{p=1}^{P} [a_p \sin^2(\pi pl/L) + b_p \sin^2(\pi pm/L)] \right| < \exp(-h_1), \tag{14}$$

where $l,m=0,1,\ldots,L-1$, $(l,m)\neq(0,0)$.

If we define $\nu_{l,m}=1-\frac{1}{P}\sum_{p=1}^{P}[a_p \sin^2(\pi pl/L) + b_p \sin^2(\pi pm/L)]$, $(l,m)\neq(0,0)$, $\nu_{\max}=\max\{\nu_{l,m}\}$, and $\nu_{\min}=\min\{\nu_{l,m}\}$ then the above stability condition becomes

$$\nu_{\max} < \exp(-h_1), \quad \nu_{\min} > -\exp(-h_1). \tag{15}$$

Performing an analysis similar to the 1D case, we obtain the expression for critical lattice size limit for nearest neighbor coupling as

$$L_{2,1} = \mathrm{Int}\left[\frac{\pi}{\sin^{-1}[\sqrt{2\tanh(h_1/2)}]}\right]. \tag{16}$$

In the case of P-neighbor diffusive coupling, we obtain the expression for the critical lattice size limit $L_{2,P}$ as

$$\frac{\sin(P\pi/L_{2,P})\cos[(P+1)\pi/L_{2,P}]}{\sin(\pi/L_{2,P})} = P\left[1 - 2\tanh(h_1/2) \times\left(1 - \frac{\sin(P\pi l_h/L_{2,P})\cos[(P+1)\pi l_h/L_{2,P}]}{P\sin(\pi l_h/L_{2,P})}\right)\right], \tag{17}$$

where $l_h=\mathrm{Int}[L_{2,1}/2]$. $L_{2,1}$ is the critical lattice size limit for two-dimensional nearest neighbor diffusively coupled map lattices and is given in Eq. (16). $L_{2,P}$ is obtained by solving Eq. (17) numerically. The dependence of $L_{2,P}$ on the maximum Lyapunov exponent of the isolated map (h_1) and the number of neighbors coupled (P) are shown in Figs. 2(a) and 2(b). In our numerical verification, we have again considered logistic map at each node. The variations of the critical coupling strength ϵ_c and the critical size limit $L_{2,P}$ (for $r=1.5$) with P are shown in Figs. 2(c) and 2(d).

IV. CONCLUSIONS

We have presented expressions for the critical lattice size limits ($L_{1,P}$ and $L_{2,P}$) for both one- and two-dimensional coupled map lattices with P-neighbor coupling. In both cases, the value of these critical size limits increase almost linearly with the number of coupled neighbors. In addition, all the above results were verified through numerical studies using coupled logistic map lattices. Moreover, as P increases to the global coupling limit we showed explicitly that the critical size limit tends to infinity. However, our results are not valid for discontinuous maps (for example, the Bernoulli map considered in Refs. [9,10]) since the linear stability analysis fails if strong nonlinear effects are present.

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